Tutorial 2

Throughout, V is a finite-dimensional inner product space.

1. Suppose $T \in \mathcal{L}(V)$ is such that for all $u, v \in V$,

$$\langle T(u), v \rangle = \langle u, T(v) \rangle$$

Let $\beta = (e_1, \dots, e_n)$ be an orthonormal basis for V and define $A = {}_\beta[T]_\beta$. What can you say about A?

2. (Bessel's Inequality) Let $\alpha = \{e_1, \dots, e_n\} \subseteq V$ be an orthonormal set. Show that for all $v \in V$,

$$\|v\|^2 \geq \sum_{k=1}^{n} |\langle v, e_k \rangle|^2$$

When do we have equality?

3. Let $(e_1, \dots, e_n)$ be an orthonormal basis for V. Show that for all $u, v \in V$,

$$\langle u, v \rangle = \sum_{k=1}^{n} \langle u, e_k \rangle \langle e_k, v \rangle$$

4. Show there exists $p \in \mathcal{P}_2(\mathbb{R})$ such that for all $f \in \mathcal{P}_2(\mathbb{R})$,

$$\int_0^1 f(x)\ dx = f(-7)p(-7) + f(1)p(1) + f(\pi)p(\pi)$$

5. A real $n \times n$ matrix A is *orthogonal* if the columns of A form an orthonormal basis for $\mathbb{R}^n$ (equipped with the standard inner product).

Show that if A is a 2×2 orthogonal matrix then there exists $\theta \in \mathbb{R}$ such that

$$A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \qquad \text{or} \qquad A = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$$

6. (6.B.12) Suppose $\langle \cdot, \cdot \rangle_1$ and $\langle \cdot, \cdot \rangle_2$ are two inner products on V with norms $\| \cdot \|_1$ and $\| \cdot \|_2$, respectively. Prove there exists a positive number c such that for all $v \in V$,

$$\|v\|_1 \leq c\|v\|_2$$

7. (a) Suppose $\mathbb{F} = \mathbb{R}$ and $v_1, \dots, v_k \in V$ are such that for all $i, j \in \{1, \dots, k\}$, $\langle v_i, v_j \rangle > 1$ when $i = j$ and $\langle v_i, v_j \rangle = 1$ when $i \neq j$. Prove $(v_1, \dots, v_k)$ is linearly independent.

   (b) Let $S_1, \dots, S_k$ be distinct subsets of $\{1, \dots, n\}$ such that for all $i, j \in \{1, \dots, n\}$ distinct, $S_i \cap S_j$ contains exactly one element. Prove $k \leq n$.